MTH 166

Lecture-5

Clairaut's Equation

Topic:

Clairaut’s Equation: A special case of equations solvable for y

Learning Outcomes:

1. How to find general solution (Not singular solution) of Clairaut’s equation.

2. How to solve equations solvable for y i.e. $y = f(x, p)$

3. How to solve equations solvable for x i.e. x$= f(y, p)$

Clairaut's Equation:

An equation of the form: $y = px \pm f(p)$ (1)

is called Clairaut's equation where $p = \frac{dy}{dx}$

General solution of equation (1) is given by:

$y = cx \pm f(c)$ (Just replace p by c in the question, we get the answer)

* It is also called as a special case of equations solvable for y: $y = f(x, p)$

** It is a very good concept for asking MCQ

Find the general solution of following differential equations:

Problem 1. $$\boldsymbol{xp^2 - yp + a = 0} \qquad \textbf{(1)}$$

Solution: $$xp^2 - yp + a = 0$$

$$\Rightarrow yp = xp^2 + a$$

$$\Rightarrow y = px + \frac{a}{p} \qquad (2)$$

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + \frac{a}{c}$ **Answer.**

Problem 2. $\boldsymbol{p = \log(px - y)}$ **(1)**

Solution: $p = \log(px - y)$

$\Rightarrow e^p = e^{\log(px-y)}$

$\Rightarrow e^p = (px - y)$

$\Rightarrow y = px - e^p$ (2)

Which is of Clairaut's form: $y = px \pm f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx - e^c$ **Answer.**

Problem 3. $\boldsymbol{y = px + \sqrt{a^2p^2 + b^2}}$ **(1)**

Solution: $y = px + \sqrt{a^2p^2 + b^2}$

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (1)

i.e. $y = cx + \sqrt{a^2c^2 + b^2}$ **Answer.**

Problem 4. $\boldsymbol{\sin px \cos y = \cos px \sin y + p}$ **(1)**

Solution: $\sin px \cos y = \cos px \sin y + p$

$\Rightarrow \sin px \cos y - \cos px \sin y = p$

$\Rightarrow \sin(px - y) = p$

$\Rightarrow (px - y) = sin^{-1}p$

$\Rightarrow y = px - sin^{-1}p$ (2)

Which is of Clairaut's form: $y = px \pm f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx - sin^{-1}c$ **Answer.**

Problem 5. $\boldsymbol{y + 2\left(\frac{dy}{dx}\right)^2 = (x+1)\left(\frac{dy}{dx}\right)}$ **(1)**

Solution: $y + 2p^2 = (x+1)p$

$\Rightarrow y = px + p - 2p^2$

$\Rightarrow y = px + p(1 - 2p)$ (2)

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + c(1 - 2c)$ **Answer.**

Problem 6. $\boldsymbol{(y - px)(p - 1) = p}$ **(1)**

Solution: $(y - px)(p - 1) = p$

$$\Rightarrow y - px = \frac{p}{p-1}$$

$$\Rightarrow y = px + \frac{p}{p-1} \qquad (2)$$

Which is of Clairaut's form: $y = px + f(p)$

So, the general solution of equation (1) is given by putting $p = c$ in equation (2)

i.e. $y = cx + \frac{c}{c-1}$ **Answer.**

Case II. Equations solvable for y. If the given equation, on solving for y, takes the form

$$y = f(x, p). \qquad \text{...(1)}$$

then differentiation with respect to x gives an equation of the form

$$p = \frac{dy}{dx} = \phi\left(x, p, \frac{dp}{dx}\right).$$

Now it may be possible to solve this new differential equation in x and p.

Let its solution be $\qquad F(x, p, c) = 0. \qquad$ ...(2)

The ellimination of p from (1) and (2) gives the required solution.

In case elimination of p is not possible, then we may solve (1) and (2) for x and y and obtain

$$x = F_1(p, c), y = F_2(p, c)$$

as the required solution, where p is the parameter.

Case III. **Equations solvable for x.** If the given equation on solving for x, takes the form

$$x = f(y, p) \qquad \text{...(1)}$$

then differentiation with respect to y gives an equation of the form

$$\frac{1}{p} = \frac{dx}{dy} = \phi\left(y, p, \frac{dp}{dy}\right)$$

Now it may be possible to solve the new differential equation in y and p. Let its solution be $F(y, p, c) = 0$.

The elimination of p from (1) and (2) gives the required solution. In case the elimination is not feasible, (1) and (2) may be expressed in terms of p and p may be regarded as a parameter.

<u>Equations solvable for y:</u> $\boldsymbol{y = f(x, p)}$

<u>Problem.</u> $\boldsymbol{y = 2px + p^n}$ **(1)**

<u>Solution:</u> $y = 2px + p^n$ (1)

Equation (1) is neither of Clairaut's form: $y = px + f(p)$ nor it can be solved for p by factorisation.

But equation (1) is of the form $y = f(x, p)$ i.e. solvable for y. So, Differentiating equation (1) w.r.t x, we get:

$$y = 2px + p^n \qquad (1)$$

$$\Rightarrow \frac{dy}{dx} = 2\left[p\frac{d}{dx}(x) + x\frac{d}{dx}(p)\right] + np^{n-1}\frac{d}{dx}(p)$$

$$\Rightarrow p = 2\left[p + x\frac{dp}{dx}\right] + np^{n-1}\frac{dp}{dx}$$

$$\Rightarrow p + 2x\frac{dp}{dx} + np^{n-1}\frac{dp}{dx} = 0$$

$$\Rightarrow p + (2x + np^{n-1})\frac{dp}{dx} = 0$$

$$\Rightarrow pdx + (2x + np^{n-1})dp = 0 \qquad (2)$$

Comparing it with: $Mdx + Ndp = 0$

Here $M = p \Rightarrow \frac{\partial M}{\partial p} = 1$

and $N = (2x + np^{n-1}) \Rightarrow \frac{\partial N}{\partial x} = 2$

Since, $\frac{\partial M}{\partial p} \neq \frac{\partial N}{\partial x}$, so, equation (2) is non-exact.

Here $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial p}}{M} = \frac{2-1}{p} = \frac{1}{p} = f(p)$

So, $I.F. = e^{\int f(p)dp} = e^{\int \frac{1}{p}dp} = e^{\log p} = p$

Multiplying equation (2) by I.F.

$[pdx + (2x + np^{n-1})dp] \times p = 0$

$\Rightarrow p^2dx + (2xp + np^n)dp = 0$

Which is an exact differential equation.

Solution: $\int_{p=con.} Mdx + \int(Terms\ of\ N\ free\ from\ x)dp = c$

$\Rightarrow \int_{p=con.} p^2dx + \int(np^n)dp = c$

$\Rightarrow p^2x + n\frac{p^{n+1}}{n+1} = c$

$$\Rightarrow x = \frac{c}{p^2} - n\frac{p^{n+1}}{n+1} \qquad (3)$$

$$\Rightarrow x = \frac{c}{p^2} - n\frac{p^{n+1}}{n+1} \qquad (3)$$

Substituting this value of x from equation (3) in equation (1), we get:

$$y = 2px + p^n$$

$$\Rightarrow y = 2p\left(\frac{c}{p^2} - n\frac{p^{n+1}}{n+1}\right) + p^n \qquad (4)$$

Thus, equations (3) and (4) taken together, in parameter p, give the general solution of equation (1).

****Similarly, we can solve equations for $\boldsymbol{x}$: $\boldsymbol{x = f(y, p)}$**

Question . Solve $y = 2px + y^2p^3$. (Bho

Solution. Given equation, on solving for x, takes the form $x = \dfrac{y - y^2p^3}{2p}$

Differentiating with respect to y, $\dfrac{dx}{dy}\left(= \dfrac{1}{p}\right) = \dfrac{1}{2} \cdot \dfrac{p\left(1 - 2y \cdot p^3 - y^2 3p^2 \dfrac{dp}{dy}\right) - (y - y^2p^3)\dfrac{dp}{dy}}{p^2}$

$$2p = p - 2yp^4 - 3y^2p^3\frac{dp}{dy} - y\frac{dp}{dy} + y^2p^3\frac{dp}{dy}$$

$$p + 2yp^4 + 2y^2p^3\frac{dp}{dy} + y\frac{dp}{dy} = 0 \text{ or } p(1 + 2yp^3) + y\frac{dp}{dy}(1 + 2yp^3) = 0.$$

$\left(p + y\dfrac{dp}{dy}\right)(1 + 2py^3) = 0$ This gives $p + y\dfrac{dp}{dy} = 0$. or $\dfrac{d}{dy}(py) = 0$.

Integrating $\quad py = c.$

Thus eliminating from the given equation and (i), we get $y = 2\dfrac{c}{y}x + \dfrac{c^3}{y^3}y^2$ or $y^2 = 2cx + c^3$

Question . *Solve* $y - 2px = \tan^{-1}(xp^2)$.

Solution. Given equation is $y = 2px + \tan^{-1}(xp^2)$

Differentiating both sides with respect to x, $\frac{dy}{dx} = p = 2\left(p + x\frac{dp}{dx}\right) + \frac{p^2 + 2xp\frac{dp}{dx}}{1 + x^2p^4}$

or $$p + 2x\frac{dp}{dx} + \left(p + 2x\frac{dp}{dx}\right) \cdot \frac{p}{1 + x^2p^4} = 0 \text{ or } \left(p + 2x\frac{dp}{dx}\right)\left(1 + \frac{p}{1 + x^2p^4}\right) = 0$$

This gives $p + 2x\, dp/dx = 0$.

Separating the variables and integrating, we have $\int \frac{dx}{x} + 2\int \frac{dp}{p} = \text{a constant}$

or $$\log x + 2\log p = \log c \quad \text{or} \quad \log xp^2 = \log c$$

whence $$xp^2 = c \qquad \text{or} \quad p = \sqrt{(c/x)}$$

Eliminating p from (*i*) and (*ii*), we get $y = 2\sqrt{(c/x)}\,x + \tan^{-1} c$